Integral soluitons of q-difference equations of the hypergeometric type with $|q| = 1$. *

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Abstract

Two integral solutions of q-difference equations of the hypergeometric type with $|q| = 1$ are constructed by using the double sine function. One is an integral of the Barnes type and the other is of the Euler type.

1 Introduction

The hypergeometric q-difference equation is one of the most important examples among q-difference systems and many studies have been achieved [2]. However, these are concerned with the case that $0 < q < 1$. In the case of $|q| = 1$, studies on q-difference systems are not sufficiently explored. The difficulty comes from the facts that fundamental functions such as " q-gamma function" are not known in the case of $|q| = 1$.

Recently, Jimbo and Miwa [3] have constructed an integral solution of the quantized Kniznik-Zamolodotikov equation with $|q| = 1$. Inspired by the result of Lukyanov [5], they have given an integral solution by means of Kurokawa's double sine function [4]. From a point of view of q-analysis, their work is very significant because it is thought of a first step of the study of q-difference system with $|q| = 1$.

In this article, we give two integral solutions of q-difference equations of the hypergeometric type with $|q| = 1$. One is an integral of the Barns type and the other is of the Euler type. Once we obtain the q-gamma function with $|q| = 1$, we can construct these integral representations in the same way as in the case

*To appear in the proceedings of the workshop "Infinite Analysis" (Oct.15–19, 1996) at the IIAS, Japan

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that $0 < q < 1$. Furthermore we can show that they are solutions of q-difference equations of the hypergeometric type with $|q| = 1$.

This article is organized as follows: In section 2, we give a survey of integral representations of the hypergeometric series and the basic hypergeometric series with $0 < q < 1$. In section 3, we define the "q-gamma function" with $|q| = 1$ by using the double sine function. In section 4, an integral of the Barnes type is introduced in the case of $|q| = 1$ and this function is shown to satisfy the hypergeometric q-difference equation. In section 5, we consider an analogue of Euler's integral representation. On this consideration, we must regard q-shifted factorials as the "q-gamma function" with $|q| = 1$, so it is needed to transform a multiplicative variable to an additive variable. This integral gives a solution of the differnce equation which is obtained by writing the hypergeometric q-difference equation by using an additive variable.

We would like to mention that our studies is significant when one considers the representation of the quantum group $SL_q(2, \boldsymbol{R})$. It is known that q must be $|q| = 1$ in $SL_q(2, \boldsymbol{R})$ (Masuda et.al. [6]), therefore, the harmonic analysis on this quantum group should be closely linked to the hypergeometric q-difference equation with $|q| = 1$.

2 Preliminaries

In this section, we give a brief survey of integral representations of the hypergeometric series

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \qquad (\text{for } |z| < 1), \tag{1}$$

where $(a)_k := a(a-1)\cdots(a-k+1)$, and of the basic hypergeometric series with $0 < q < 1$

$$\phi(q^a, q^b, q^c; q, z) = \sum_{k=0}^{\infty} \frac{(q^a; q)_k (q^b; q)_k}{(q^c; q)_k (q; q)_k} z^k \qquad (\text{for } |z| < 1), \tag{2}$$

nnn where $(a; q)_k := \prod_{l=0}^{k} (1 - aq^l)$.

2.1 Barnes' contour integral representation

Barnes' contour integral representation is so defined that sum of residue of the integrand is equal to the hypergeometric series.

Let us define $(-z)^s := \exp(s \log(-z))$, where we choose such a branch of logarithm that this logarithm takes real value when z is on negative real line. To define this integral, the following lemma is important.

Lemma 2.1 (1) *The function $\pi(-z)^s/\sin\pi s$ has simple poles at $s = k$ ($k \in \boldsymbol{Z}$), and the residue there is z^k.*
(2). *We have, for $|z| < 1$, that*

$$\frac{\pi(-z)^s}{\sin\pi s} = O\left[\exp\left\{-|\Im s|\arg(-z)\right\}\right] \tag{3}$$

as $\Im s \to \infty$ preserving $|\Re s| < \infty$.

Let us fix a real number δ such that $0 < \delta < \pi$ and suppose z to be in a sector $S_1 := \{z \in \boldsymbol{C} | -\pi + \delta < \arg(-z) < \pi - \delta, |z| < 1\}$. Barnes' contour integral of the hypergeometric series is given as follows:

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\left(\frac{-1}{2\pi i}\right)\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(s+1)}\frac{\pi(-z)^s}{\sin\pi s}ds \tag{4}$$

where the contour lies on the right of poles

$$s = -an + n \qquad s = -b + n \qquad (n \in \boldsymbol{Z}_{\leq 0}) \tag{5}$$

and on the left of poles $s = m$ ($m \in \boldsymbol{Z}_{\geq 0}$).

Thanks to the Stirling formula of the gamma function and Lemma 2.1 (2), we can see that the integral (4) converges uniformly in S_1 . Furthermore, by using deformation of the integral contour and residue calculus based on Lemma 2.1 (1), one can show that the integral (4) is the hypergeometric series. For the details, see [9].

Next, we consider a q-analogue of (4) in the case that $0 < q < 1$. Let us put $q = e^{-2\pi\tau}$ ($\tau > 0$). The counterpart of (4), which is known as Watson's contour integral, is given as follows:

$$\phi(q^a, q^b, q^c; q, z) = \frac{\Gamma(c;q)}{\Gamma(a;q)\Gamma(b;q)}\left(\frac{-1}{2\pi i}\right)\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s;q)\Gamma(b+s;q)}{\Gamma(c+s;q)\Gamma(s+1;q)}\frac{\pi(-z)^s}{\sin\pi s}ds \tag{6}$$

where $\Gamma(z;q)$ is the q-gamma function defined by

$$\Gamma(z:q) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \tag{7}$$

and the contour lies on the right of poles

$$s = -a + n_1 + \frac{n_2}{\tau}, \qquad s = -b + n_1 + \frac{n_2}{\tau}, \qquad (n_1 \in \boldsymbol{Z}_{\leq 0}, \quad n_2 \in \boldsymbol{Z})$$

and on the left of poles $s = m$ ($m \in \boldsymbol{Z}_{\geq 0}$).

From Lemma 2.1 (2) and the fact that

$$\left|\frac{\Gamma(a+s;q)\Gamma(b+s;q)}{\Gamma(c+s;q)\Gamma(1+s;q)}\right| \leq \text{Const.}\prod_{k=1}^{\infty}\frac{(1+e^{-(c+k+\Re s)\tau})(1+e^{-(1+k+\Re s)\tau})}{(1-e^{-(a+k+\Re s)\tau})(1-e^{-(b+k+\Re s)\tau})},$$

it follows that the integral (6) converges uniformly in S_1. By using the same technique, one can show that the integral (6) is equal to the basic hypergeometric series [2].

2.2 Euler's integral representation

Euler's integral representation for the hypergeometric series is

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt. \tag{8}$$

From the binomial theorem and an integral representation of the beta function, it follows that the integral (8) gives the hypergeometric series.

A q-analogue of this representation is given, by using the Jackson integral, as follows:

$$\phi(q^a, q^b, q^c; q, z) = \frac{\Gamma(c;q)}{\Gamma(b;q)\Gamma(c-b;q)} \int_0^1 t^b \frac{(tzq^a;q)_\infty (tq;q)_\infty}{(tz;q)_\infty (tq^{c-b};q)_\infty} \frac{d_q t}{t}. \tag{9}$$

In the same way as the classical case, by using the q-binomial theorem and the Jackson integral representation of the q-beta function, we can prove that (9) is equal to the basic hypergeometric series.

2.3 The hypergeometric q-difference equations

When $|q| < 1$, the basic hypergeometric series $\phi(q^a, q^b, q^c; q, z)$ is convergent for $|z| < 1$, and satisfies the hypergeometric q-difference equation;

$$(L_q \phi)(z) = 0, \tag{10}$$

where

$$\begin{aligned}
&[z] := \frac{1-q^z}{1-q}, \qquad (T_q f)(z) := f(qz), \\
&D_q := \frac{1-T_q}{(1-q)z}, \qquad [\vartheta + a] := \frac{1-q^a T_q}{1-q} \\
&L_q := z^{-1}[\vartheta][\vartheta + c - 1] - [\vartheta + a][\vartheta + b] \\
&\quad\;\; = z(q^c - q^{a+b+1} z) D_q^2 \\
&\qquad - \left\{ [c] - \frac{(1-q^a)(1-q^b) - (1-q^{a+b+1})}{1-q} z \right\} D_q - [a][b].
\end{aligned} \tag{11}$$

We should note that the basic hypergeometric series with $|q| = 1$ is not convergent (so it gives only a formal solution to the hypergeometric q-difference equation).

3 "q-gamma funciton" with $|q| = 1$

Let us define a function $\widetilde{\Gamma}(z;q)$ which satisfies

$$\widetilde{\Gamma}(z+1;q) = [z]\widetilde{\Gamma}(z;q) \tag{12}$$

in the case of $|q| = 1$.

For this end, we need *the double zeta function* $\zeta_2(s, z|\boldsymbol{\omega})$, *the double gamma function* $\Gamma_2(z|\boldsymbol{\omega})$ and *the double sine function* $S_2(z|\boldsymbol{\omega})$ (cf. [1], [3], [4], [8]).

Definition 3.1 *For* $\boldsymbol{\omega} := (\omega_1, \omega_2) \in \boldsymbol{C}^2$*, we define* $\zeta_2(s, z|\boldsymbol{\omega})$*,* $\Gamma_2(z|\boldsymbol{\omega})$ *and* $S_2(z|\boldsymbol{\omega})$ *by*

$$\begin{aligned}
\zeta_2(s, z|\boldsymbol{\omega}) &:= \sum_{m_1, m_2 \in \boldsymbol{Z}_{\geq 0}} (z + m_1\omega_1 + m_2\omega_2)^{-s}\,, \\
\Gamma_2(z|\boldsymbol{\omega}) &:= \exp\left(\frac{\partial}{\partial s}\zeta_2(s, z|\boldsymbol{\omega})|_{s=0}\right), \\
S_2(z|\boldsymbol{\omega}) &:= \Gamma_2(z|\boldsymbol{\omega})^{-1}\Gamma_2(\omega_1 + \omega_2 - z|\boldsymbol{\omega}).
\end{aligned}$$

It is known that the double sine function satisfies the functional relation

$$\frac{S_2(z + \omega_1|\boldsymbol{\omega})}{S_2(z|\boldsymbol{\omega})} = \frac{1}{2\sin\frac{\pi z}{\omega_2}}. \tag{13}$$

Thus, we can construct a function satisfying (12) by using $S_2(z|\boldsymbol{\omega})$. We suppose that $|q| = 1$ and that q is not a root of unity. Let us put $q = e^{2\pi i\omega}$ $(0 < \omega < 1, \omega \notin \boldsymbol{Q})$.

Definition 3.2 *We set*

$$\widetilde{\Gamma}(z; q) := (q - 1)^{1-z} i^{z-1} q^{\frac{z(z-1)}{4}} S_2(z|(1, \frac{1}{\omega}))^{-1}, \tag{14}$$

which has the following properties.

Proposition 3.3 (1) $\widetilde{\Gamma}(z; q)$ *has simple zeros at* $z = n_1 + \frac{n_2}{\omega}$ $(n_1, n_2 \in \boldsymbol{Z}_{>0})$*, and has simple poles at* $z = n_1 + \frac{n_2}{\omega}$ $(n_1, n_2 \in \boldsymbol{Z}_{\leq 0})$*.*
(2) $\widetilde{\Gamma}(z; q)$ *satisfies the functional relation (12).*
(3) *If we take* $z \to \infty$ *as* z *is in any sector not containing real line then* $\widetilde{\Gamma}(z; q)$ *has the following asymptotic behavior.*

$$\begin{aligned}
\widetilde{\Gamma}(z; q) = \exp\big[&(1 - z)\log(q - 1) + (z - 1)\log i \\
&+ \frac{z(z-1)}{4}\log q \mp \pi i\left\{\frac{\omega z^2}{2} - \frac{\omega + 1}{2}z\right\} + O(1)\big] \quad (for \pm \Im z > 0).
\end{aligned}$$

These properties follow from the facts in the papers [3], [8].

Remark 3.4 *In the case that* $0 < q < 1$*, we can also define* $\widetilde{\Gamma}(z, q)$ *by Definition 3.2 (in this case,* $\omega = it$*,* $t > 0$*). Of course, we can see that* $\widetilde{\Gamma}(z, q) = C(z, q)\Gamma(z, q)$*, where* $C(z, q)$ *is a function satisfying* $C(z + 1, q) = C(z, q)$ *(cf. [8]).*

4 An integral representation of the Barnes type with $|q| = 1$

4.1 Definition of the integral

In order that the integral makes sense, we impose some conditions on parameters a, b and c.

Conditions on the parameters $(B1)$ *If we define sets* A_1 *and* A_2 *of the parameters by* $A_1 := \{a, b\}$, $A_2 := \{c, 1\}$, *then we suppose that*

$$''\Re\alpha > \Re\beta \qquad \text{for} \quad \forall\alpha \in A_1, \quad \forall\beta \in A_2''$$

or

$$''\Im\alpha \neq \Im\beta \qquad \text{for} \quad \forall\alpha \in A_1, \quad \forall\beta \in A_2''$$

$(B2)$ *We suppose that*

$$\omega\Re(a + b - c + 1) < 1.$$

Under these conditions, we can define a function $\Phi(a, b, c; q, z)$ in the same way as Bernes' contour integral by using $\widetilde{\Gamma}(z, q)$.

Definition 4.1 *Let us fix a real number* δ *such that* $0 < \delta < \pi - \pi\omega\Re(a + b - c + 1)$. *For* z *in the sector* $S := \{z \in \boldsymbol{C} | -\pi + \delta < \arg(-z) < \pi - 2\pi\omega\Re(a + b - c + 1), |z| < 1\}$, *we define* $\varphi(a, b, c; q; s, z)$ *and* $\Phi(a, b, c; q, z)$ *by*

$$\varphi(a, b, c; q; s, z) := \frac{\widetilde{\Gamma}(a + s; q)\widetilde{\Gamma}(b + s; q)}{\widetilde{\Gamma}(c + s; q)\widetilde{\Gamma}(1 + s; q)} \frac{\pi(-z)^s}{\sin \pi s}$$

$$\Phi(a, b, c; q, z) := \frac{\widetilde{\Gamma}(c; q)}{\widetilde{\Gamma}(a; q)\widetilde{\Gamma}(b; q)} \left(\frac{-1}{2\pi i}\right) \int_{-i\infty}^{i\infty} \varphi(a, b, c : q; s, z) ds \qquad (15)$$

where the contour lies on the right of poles

$$s = -a + n_1 + \frac{n_2}{\omega}, \quad s = -b + n_1 + \frac{n_2}{\omega} \quad (n_1, n_2 \in \boldsymbol{Z}_{\leq 0})$$

and on the left of poles

$$s = -c + n_1 + \frac{n_2}{\omega}, \quad s = -1 + n_1 + \frac{n_2}{\omega} \quad (n_1, n_2 \in \boldsymbol{Z}_{>0}),$$
$$s = m \quad (m \in \boldsymbol{Z}_{\geq 0}).$$

By using Lemma 2.1 (2) and Proposition 3.3 (2), it is shown that

$$\varphi(a, b, c; q; s, z) = O\left[\exp(-\delta|s|)\right] \quad \text{as} \quad s \to \pm i\infty$$

under the condition (B1). Thus the integral (15) converges uniformly in S, and the analytic continuation (also denote $\Phi(a, b, c; q, z)$) defines a many-valued analytic function of z.

4.2 The hypergeometric q-difference equation with $|q| = 1$

We prove that $\Phi(a, b, c; q, z)$ is a solution of the hypergeometric q-difference equation with $|q| = 1$. We also use the notation (10) in the case that $|q| = 1$.

Theorem 4.2 $\Phi(z) := \Phi(a, b, c; q, z)$ *satisfies the hypergeometric q-difference equation*

$$(L_q \Phi)(z) = 0.$$

Outline of Proof: From the condition (B2), it follows that the action of L_q commute with the integration. On the other hand, straightfoward calculation shows that the integrand $\varphi(a, b, c; q; s, z)$ satisfies the relation

$$(L_q \varphi)(a, b, c; q; s, z) = \varphi(a+1, b+1, c; q; s-1, z) - \varphi(a+1, b+1, c; q; s, z). \quad (16)$$

By means of Cauchy's theorem, one can verify that the integral of the right-hand side of (16) vanishes. ∎

5 An integral representation of the Euler type with $|q| = 1$

5.1 Definition of the integral

First let us recall Euler's integral (9) in the case that $0 < q < 1$. If we transform the variables z and t in the integrand of (9) to q^x and q^s respectively, then we have

$$\text{(the integrand of (9))} = \text{ Const. } \frac{\Gamma(s + x; q)\Gamma(s + c - b; q)}{\Gamma(s + x + a; q)\Gamma(s + 1; q)} q^{bs}.$$

Therefore, in the case that $|q| = 1$, we consider the integral

$$\int \frac{\widetilde{\Gamma}(s + x; q)\widetilde{\Gamma}(s + c - b; q)}{\widetilde{\Gamma}(s + x + a; q)\widetilde{\Gamma}(s + 1; q)} q^{bs} ds \quad (17)$$

as a counterpart of (9). In order that the integral makes sense, we impose the following conditions on the parameters a, b and c.

Conditions on the parameters. $(E1)$ $b - c \notin \boldsymbol{R}_{>0}$, $(E2)$ $a \notin \boldsymbol{R}_{<0}$, $(E3)$ $\Re b > 0$, $\Re(a - c - 1) > 0$.

Under these conditions, we can take such a suitable contour that the integral (17) makes sense.

Definition 5.1 *For $x \notin \boldsymbol{R}_{<0}$, we define a function $\Psi(a,b,c;q,x)$ by*

$$\Psi(a,b,c;q,x) := \int_{-i\infty}^{i\infty} \frac{\widetilde{\Gamma}(s+x;q)\widetilde{\Gamma}(s+c-b;q)}{\widetilde{\Gamma}(s+x+a;q)\widetilde{\Gamma}(s+1;q)} q^{bs} ds \tag{18}$$

where the contour lies on the right of the poles

$$s = -x + n_1 + \frac{n_2}{\omega}, \quad s = b - c + n_1 + \frac{n_2}{\omega}, \quad (n_1, n_2 \in \boldsymbol{Z}_{\leq 0}),$$

and on the left of the poles

$$s = -x - a + n_1 + \frac{n_2}{\omega}, \quad s = -1 + n_1 + \frac{n_2}{\omega}, \quad (n_1, n_2 \in \boldsymbol{Z}_{>0}).$$

Thanks to the conditions (E1) and (E2), we can take the contour of Definition 5.1. From the condition (E3), it follows that the integral (18) converges uniformly and defines a single-valued analytic function of x.

5.2 The difference equation for $\Psi(a,b,c;q,x)$

Let us present an equation which $\Psi(a,b,c;q,x)$ satisfies. For this end, we write the hypergeometric q-difference equation by using the "additive" variable x. We employ the following notatiaons;

$$(T_+g)(x) := g(x+1), \quad [\vartheta + a]_+ := \frac{1 - q^a T_+}{1-q}$$

$$\begin{aligned} L_+ &:= q^{-x}[\vartheta]_+[\vartheta + c - 1]_+ - [\vartheta + a]_+[\vartheta + b]_+ \\ &= \frac{1}{(1-q)^2} \Big[(q^{c-1-x} - q^{a+b}) \left\{ T_+^2 - (1+q)T_+ + q \right\} \\ &\quad - \left\{ (1-q^c)q^{-x} + (1-q^a)(1-q^b) - (1-q^{a+b+1}) \right\} (T_+ - 1) \\ &\quad -(1-q^a)(1-q^b) \Big] \end{aligned}$$

Then the next theorem holds.

Theorem 5.2 $\Psi(x) := \Psi(a,b,c;q,x)$ *satisfies the difference equation*

$$(L_+\Psi)(x) = 0.$$

This theorem can be proved just like Theorem 5.1.

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